A TOPOLOGY ON E-THEORY

JOINT WORK WITH C. SCHAFHAUSER

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The E-theory group $E(A, B)$

- bifunctor from sep. C^*-algebras to abelian groups, defined by Connes–Higson (1990)
- stable, homotopy invariant
- has composition product $E(A, B) \times E(B, C) \to E(A, C)$

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Similar to $KK(A, B)$, but:

- is half-exact; has 6-term exact sequences in each variable (only true for KK under nuclearity assumption)

Asymptotic morphisms $\varphi\colon A \xrightarrow{\approx} B$

- family $(\varphi_t)_{t\in[0,\infty)}$ of functions $A \to B$
- $t \mapsto \varphi_t(a)$ is continuous $\forall a \in A$
- asymptotically satisfies properties of *-homomorphisms ($\|\varphi_t(a+\lambda b) - \varphi_t(a) - \lambda\varphi_t(b)\| \to 0$, etc.)

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Alternative point of view

Can identify an asymptotic morphism $\varphi \colon A \xrightarrow{\approx} B$ with a $*$-hom

$$\varphi_{as} \colon A \to \frac{C_b\big([0,\infty), B\big)}{C_0\big([0,\infty), B\big)}.$$

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Definition

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- $E(A, B) := [[SA \otimes \mathcal{K}, SB \otimes \mathcal{K}]]$

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Fact: $KK(A, B) \cong E(A, B)$ if A is nuclear.

Some history

- references to the topology on $KK(A, B)$ go back to Brown–Douglas–Fillmore ('77)
- was first studied in depth by L. Brown ('84), then Salinas ('92), in connection with quasidiagonality
- further developed by Schochet ('01), before being defined in general by Pimsner (unpublished) and Dadarlat ('05)

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The different definitions use representation-theoretic pictures of $KK(A, B)$, which aren't available for $E(A, B)$.

The “Hausdorffization” of $KK(A, B)$: $KL(A, B)$

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- important in classification theory of nuclear C^*-algebras; used to obtain uniqueness for $*$-hom's
- Lin: unital $*$-hom's φ, ψ between Kirchberg algebras are approximately unitarily equivalent $\Leftrightarrow KL(\varphi) = KL(\psi)$
- KL captures the limiting behavior of the controlled KK-theory groups of Willett and Yu

Theorem (C–Schafhauser)

There is a unique second-countable topology on $E(A,B)$ s.t.

$$x_n \to x \in E(A,B) \quad \Leftrightarrow \quad \begin{array}{l} \exists\, y \in E\big(A, C(\mathbb{N}^\dagger, B)\big) \text{ s.t.} \\ y(n) = x_n \ \text{ and } \ y(\infty) = x \ \ \forall n \in \mathbb{N} \end{array}$$

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(known as "Pimsner's condition").

Moreover, the product $E(A,B) \times E(B,C) \to E(A,C)$ is continous (just like for KK).

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Theorem (C–Schafhauser)

$$EL(\varinjlim A_n, B) \cong \varprojlim EL(A_n, B)$$

so

$$KL(\varinjlim A_n, B) \cong \varprojlim KL(A_n, B)$$

when all A_n are nuclear.

- Recall: $[[A, B]] \coloneqq \{A \xrightarrow{\approx} B\}/\text{homotopy}.$

METHODS USED

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- Recall: $[[A, B]] := \{A \xrightarrow{\approx} B\}/\text{homotopy}$.
- We study $[[A, B]]$ and define topology on it. Topology on $E(A, B)$ is special case.
- Define and study $[[A, B]]_{\mathrm{Hd}}$, the "Hausdorffization" of $[[A, B]]$. $EL(A, B)$ is special case.
- Main tool: Blackadar's shape theory and its connection with asymptotic morphisms, discovered by Dadarlat.

- General idea: write a C^*-algebra A as a direct limit

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \to A$$

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- "Nice": *semiprojective*, an analog of ANR for topological spaces.
- Not known if every C^*-algebra can be written as a limit of semiprojective C^*-algebras.
- *However:* every C^*-algebra can be written as such a limit with semiprojective connecting maps (a *shape system*).

The shape category SH of Blackadar–Dadarlat

Objects: shape systems (A_n, α_n)

Morphisms:

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft_h"] & A_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft_h"] & A_{n+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & B_{n-1} \arrow[r] & B_n \arrow[r] & B_{n+1} \arrow[r] & \cdots
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Getting $\varphi \colon A \xrightarrow{\approx} B$ from a morphism in SH:

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A_1 \arrow[rr, "\alpha_1"] \arrow[dr, "h_1"] \arrow[dd, "\varphi_1"] & & A_2 \arrow[rr, "\alpha_2"] \arrow[dr, "h_2"] \arrow[dd, "\varphi_2"] & & A_3 \arrow[r] \arrow[dd, "\varphi_3"] & \cdots \arrow[r] & A \\
& C([0,1], B_2) \arrow[dr, shift left, "\mathrm{ev}_1"] \arrow[dr, shift right, "\mathrm{ev}_0"'] & & C([0,1], B_3) \arrow[dr, shift left, "\mathrm{ev}_1"] \arrow[dr, shift right, "\mathrm{ev}_0"'] & & & \\
B_1 \arrow[rr, "\beta_1"'] & & B_2 \arrow[rr, "\beta_2"'] & & B_3 \arrow[r] & \cdots \arrow[r] & B
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& C([0,1], B_2) \arrow[dr, shift left, "\mathrm{ev}_1"] \arrow[dr, shift right, "\mathrm{ev}_0"'] & & C([1,2], B_3) \arrow[dr, shift left, "\mathrm{ev}_1"] \arrow[dr, shift right, "\mathrm{ev}_0"'] & & & \\
B_1 \arrow[rr, "\beta_1"'] & & B_2 \arrow[rr, "\beta_2"'] & & B_3 \arrow[r] & \cdots \arrow[r] & B
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Theorem (Dadarlat)

Every asymptotic morphism $\varphi : A \xrightarrow{\approx} B$ *arises from a morphism of shape systems.*

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There is an equivalence of categories $\mathsf{SH} \xrightarrow{\sim} \mathsf{AM}_{\mathrm{Hd}}$, where $\mathsf{AM}_{\mathrm{Hd}}$ is the category whose objects are separable C^*-algebras and where the morphisms between A and B are the elements of $[[A, B]]_{\mathrm{Hd}}$.

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A, B separable C^*-algebras: $\quad A \underset{E}{\sim} B \Leftrightarrow A \underset{EL}{\sim} B.$

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A, B separable C^*-algebras: $\quad A \underset{E}{\sim} B \Leftrightarrow A \underset{EL}{\sim} B.$

In particular, if A, B nuclear: $\quad A \underset{KK}{\sim} B \Leftrightarrow A \underset{KL}{\sim} B.$

THANK YOU!